Published by Institute of Physics Publishing for SISSA

Received: *December 12, 2007*
Revised: *January 22, 2008*
Accepted: *February 1, 2008*
Published: *February 22, 2008*

Fusion of conformal interfaces

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Abstract: We study the fusion of conformal interfaces in the $c = 1$ conformal field theory. We uncover an elegant structure reminiscent of that of black holes in supersymmetric theories. The role of the BPS black holes is played by topological interfaces, which (a) minimize the entropy function, (b) fix through an attractor mechanism one or both of the bulk radii, and (c) are (marginally) stable under splitting. One significant difference is that the conserved charges are logarithms of natural numbers, rather than vectors in a charge lattice, as for BPS states. Besides potential applications to condensed-matter physics and number theory, these results point to the existence of large solution-generating algebras in string theory.

Keywords: Black Holes in String Theory, D-branes, Conformal Field Models in String Theory.

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JHEP02(2008)085

Contents

1. Introduction

Conformal interfaces in two dimensions [1] are scale invariant junctions of two conformal field theories. They are generalizations of conformal defects and of conformal boundaries which correspond, respectively, to the case of identical theories on the two sides,[1] or of a trivial theory (with no massless degrees of freedom) on one side. There is an extensive literature, and many beautiful experimental realizations of such objects in condensed-matter physics (for reviews and many references see for instance [2, 3]). Applications to condensed-matter physics are outside our scope in this work.

Two or more interfaces between the same pair of theories can be added. This amounts to endowing them with a finite-dimensional space of (Chan-Patton or "quantum-dot") degrees of freedom. Furthermore, an interface between CFT_1 and CFT_2, and an interface between CFT_2 and CFT_3 can, in principle, be fused to produce a $\mathrm{CFT}_1 \rightarrow \mathrm{CFT}_3$ interface.

[1] In the literature, general interfaces are sometimes also referred to as defects. We believe it is useful to distinguish the two, and not only for semantic reasons. Defects live at a given point in the moduli space of CFTs, and can be always multiplied together. General interfaces, on the other hand, are intertwiners between different CFTs.

The process is in general singular, because fusion (or its inverse, dissociation) corresponds to non-trivial renormalization-group flow. An exception to this rule occurs when one of the interfaces transmits all incident energy, in which case the left- and right- Virasoro charges are separately conserved. Interfaces of this type, first introduced by Petkova and Zuber [4], can move freely on a Riemann surface and are, in this sense, "topological". Their fusion is non-singular. Many examples of conformal and topological interfaces have been worked out in the literature over the past few years (a list of references is [5-18]). Lifts to topological gauge theories in higher dimensions [19-22], and dual holographic interpretations [1, 23-27] have been also analyzed.[2]

One of the most interesting aspects of topological interfaces is the fact that they are universal maps transforming one set of D-branes into another [7, 10]. All the symmetry transformations of a CFT can be, in particular, implemented in this fashion [9]. A generic topological interface does not, however, correspond always to a symmetry: its action changes the mass, charges and other properties of the D-branes, and possibly even those of the bulk geometry. This makes it tempting to speculate [29] that the algebra of all conformal interfaces is a solution-generating algebra of string theory, similar to the Ehlers-Geroch transformations of General Relativity. A classical-geometric intepretation for this algebra has been suggested in ref. [10]. It is based on the folding trick [30, 1], which identifies an interface with a middle-dimensional brane in the product target space $M_1 \times M_2$. Such a brane can be described, at least locally, in terms of a multiple embedding of M_2 into M_1.[3] This embedding determines the image of M_2, and of all its D-brane submanifolds, under the interface map.

A crucial question is whether this story survives quantization, and in particular the singularities of interface fusion. In this paper we will answer the question in the simplest setting, that of the $c = 1$ conformal field theory. The boundary states of this model are classified [31], its topological interfaces have been studied [16], and most calculations can be done explicitly. Our analysis will confirm the existence of a conformal-interface algebra, and its geometric interpretation in the classical limit. At the same time, a beautiful and unexpected picture will emerge: the topological interfaces of this simple model behave in many ways like BPS black holes! They are minima of an entropy function, they freeze by an attractor mechanism [32] one or both of the bulk radii, and they are stable against decay to more elementary interfaces. Their algebra is reminiscent of the Harvey-Moore algebra of BPS states [33]. There is, however, one significant difference: the conserved charges of these topological interfaces do not take values in a regular lattice, but they are instead the logarithms of integers. A quantum gas of such particles had been imagined in the past by Julia [34] in an effort to rephrase the Riemann hypothesis as a problem in statistical mechanics.

Supersymmetry plays no role in our discussion here. A different line of approach, that avoids the problem of singularities, has been to study the fusion of defect lines in theories

[2]For an entry into the extensive literature on superconformal defects and the AdS/CFT correspondence in $d = 4$ we refer the reader to the review [28].

[3]Assuming for simplicity that the world-volume gauge fields are zero.

with extended supersymmetry by twisting to a topological theory, see [21, 22] for results on $N = 4$ gauge theories in four dimensions, and [17] for $N = 2$ theories in two dimensions.

The structure of our paper is as follows: In section 2 we define our conventions, and review the boundary states for toroidally-compactified free-boson CFTs. In section 3 we describe the unfolding of the $\mathrm{U}(1)^2$ symmetric boundary states of the two-scalar theory to intertwining operators acting on the moduli space of circle compactifications. We explain the special role of topological interfaces, and point out the analogy with BPS black holes. Sections 4 and 5 contain our main results. We show there that the fusion of two symmetric interfaces is well-defined, and that it does not depend on the radius of the collapsed region. This reduces the calculation of the algebra to the topological case, studied in ref. [16]. We explain why the integer interface charges are multiplicatively conserved, and discuss interface stability in a way reminiscent again of black holes. Finally, in section 6 we extend the discussion to topological interfaces for which all CFT moduli are completely fixed, and which have no semiclassical limit. The operator that interpolates between the circle and orbifold branches is of this type. A detailed analysis of the extended $c = 1$ interface algebra is postponed to future work.

2. Boundary states of toroidal CFT

2.1 Dirichlet and Neumann states

We will use the boundary-state formalism [35, 36] in which boundary conditions are described by states in the Hilbert space of the bulk CFT. Let us start by recalling the expressions of the boundary states for a free scalar field compactified on a circle of radius R. The mode expansion of the field on the cylinder, parametrized by $\sigma \in [0, 2\pi)$ and τ, is given by

$$\phi(\tau,\sigma) = \hat{\phi}_0 + \frac{\hat{N}}{2R}\tau + \hat{M}R\sigma + \sum_{n=1}^{\infty} \frac{i}{2\sqrt{n}} \left(a_n e^{-in(\tau+\sigma)} + \tilde{a}_n e^{-in(\tau-\sigma)} - h.c. \right) , \qquad (2.1)$$

where $\hat{N}$, $\hat{M}$ are the integer-valued momentum and winding operators, and $h.c.$ denotes the hermitean conjugate. The canonical commutation relations imply

$$[a_n, a_m^\dagger] = [\tilde{a}_n, \tilde{a}_m^\dagger] = \delta_{n,m} \quad \text{and} \quad [\hat{\phi}_0, \frac{\hat{N}}{R}] = i\,, \qquad (2.2)$$

while the Hamiltonian reads

$$H = L_0 + \tilde{L}_0 = \frac{\hat{N}^2}{4R^2} + \hat{M}^2 R^2 + \sum_{n=1}^{\infty} n(a_n^\dagger a_n + \tilde{a}_n^\dagger \tilde{a}_n) - \frac{1}{12}\ . \qquad (2.3)$$

The two simplest boundary states of this theory[4] correspond to the Dirichlet and

[4]The free-boson theory contains also boundary states that break all $\mathrm{U}(1)$ symmetries of the bulk [31]. We will discuss these in section 6.

Neumann boundary conditions for ϕ. They are given by

$$\underline{\text{Dirichlet}}: \qquad \|\text{D0}\,\rangle\!\rangle \;=\; \prod_{n=1}^{\infty} \exp(a_n^\dagger \tilde{a}_n^\dagger)\,\Big(\frac{1}{\sqrt{2R}} \sum_{N=-\infty}^{\infty} e^{-i\frac{N}{R}\phi_0} |N,0\rangle\Big) \tag{2.4}$$

$$\underline{\text{Neumann}}: \qquad \|\text{D1}\rangle\!\rangle \;=\; \prod_{n=1}^{\infty} \exp(-a_n^\dagger \tilde{a}_n^\dagger)\,\Big(\sqrt{R} \sum_{M=-\infty}^{\infty} e^{iM\tilde{\phi}_0} |0,M\rangle\Big) \tag{2.5}$$

where $|N,M\rangle$ is the normalized ground state in a given momentum and winding sector. Using the commutation relations one verifies easily that

$$\phi(0,\sigma)\|\text{D0}\,\rangle\!\rangle \;=\; \phi_0\|\text{D0}\,\rangle\!\rangle \quad \text{and} \quad \partial_\tau\phi(0,\sigma)\|\text{D1}\rangle\!\rangle \;=\; 0 \qquad \forall\sigma\,, \tag{2.6}$$

as claimed. The arbitrary parameters ϕ_0 and $\tilde{\phi}_0$ are, respectively, the position of the D0 brane, and the dual Wilson line on the D1 brane (normalized so as to be periodic under 2π shifts).

The boundary conditions, eq. (2.6), do not determine the normalization of the corresponding states. This is usually fixed by Cardy's condition [37], i.e. by the requirement that the annulus diagram be equal to the finite-temperature partition function in the transverse channel. Although the result for the case at hand is known, it will be useful to work it out explicitly. Considering for instance the D0 brane, we may factorize the annulus diagram as follows:

$$A_{\text{DD}} \equiv \langle\!\langle \text{D0}\|\, q^H\, \|\text{D0}\,\rangle\!\rangle \;=\; q^{-\frac{1}{12}}\, \langle\phi_0|q^H|\phi_0\rangle \prod_{n=1}^{\infty} \langle 0|e^{q^{2n}a\tilde{a}} e^{a^\dagger\tilde{a}^\dagger}|0\rangle\,, \tag{2.7}$$

where $q = e^{-T}$, $|\phi_0\rangle$ is the state within the parentheses in equation (2.4), i.e. the ground state for fixed value of $\hat{\phi}_0$, the a and $\tilde{a}$ are canonically normalized lowering operators of a double harmonic-oscillator system (the same for all values of n), and $|0\rangle$ is the harmonic-oscillator ground state. To calculate the individual matrix elements we use the operator identities

$$e^{AB} = \int \frac{d^2z}{\pi}\, e^{-z\bar{z}-zA-\bar{z}B} \qquad \text{if} \quad [A,B]=0\,, \tag{2.8}$$

and $e^Ae^B = e^Be^Ae^{[A,B]}$ if $[A,B]$ is a c-number. A simple calculation then gives:

$$\begin{aligned}\langle 0|e^{q^{2n}a\tilde{a}} e^{a^\dagger\tilde{a}^\dagger}|0\rangle \;&=\; \int \frac{d^2z d^2w}{\pi^2}\, e^{-z\bar{z}-w\bar{w}} \langle 0|e^{-zq^n a-\bar{z}q^n\tilde{a}} e^{-wa^\dagger-\bar{w}\tilde{a}^\dagger}|0\rangle \\ &=\; \int \frac{d^2z d^2w}{\pi^2}\, e^{-z\bar{z}-w\bar{w}} e^{q^n(zw+\bar{z}\bar{w})} \;=\; \frac{1}{1-q^{2n}}\,.\end{aligned} \tag{2.9}$$

Computing the remaining matrix element, and combining everything leads to

$$A_{\text{DD}} = \Big(\frac{1}{2R} \sum_{N=-\infty}^{\infty} q^{\frac{N^2}{4R^2}}\Big)\,\frac{1}{\eta(q^2)} \;=\; \Big(\sum_{M=-\infty}^{\infty} \widetilde{q}^{\;4M^2R^2}\Big)\,\frac{1}{\eta(\widetilde{q}^{\;2})}\,, \tag{2.10}$$

where η is the Dedekind function and $\widetilde{q} = e^{-\pi^2/T}$. The second equality follows from the modular properties of η and the Poisson resummation formula. The final expression is a partition function with integer non-negative multiplicities, and a unique lowest-energy state. This shows that the D0 boundary state has been normalized consistently, and that it describes an elementary brane.

2.2 Branes in the two-scalar model

Let us next consider two scalar fields, ϕ^1 and ϕ^2, compactified on two circles with radii R_1 and R_2. Taking the tensor product of the states (2.4) and (2.5) gives the four *factorizable* branes of the theory, which correspond to independent Neumann or Dirichlet conditions for each scalar. Put differently, these describe a D0, a D2 and two D1 branes, with the latter wrapping the two elementary cycles of the (ϕ^1, ϕ^2) torus. The most general elementary D1-brane winds (k_1, k_2) times around these cycles, where k_1 and k_2 may be assumed relatively prime and k_1 positive. The corresponding boundary states can be constructed easily starting with the factorizable $(1, 0)$ brane and then rotating by an angle

$$\vartheta = \tan^{-1}\left(\frac{k_2 R_2}{k_1 R_1}\right) . \tag{2.11}$$

The result reads

$$\| \mathrm{D1}, \vartheta \rangle\rangle \;=\; \prod_{n=1}^{\infty} (e^{S^{(+)}_{ij} a^i_n \tilde{a}^j_n})^{\dagger} \left(g^{(+)} \sum_{N,M=-\infty}^{\infty} e^{iN\alpha - iM\beta} |k_2 N, k_1 M\rangle \otimes | - k_1 N, k_2 M\rangle \right) \tag{2.12}$$

where α and β are position and Wilson-line moduli, the ground states in the tensor product correspond to ϕ^1 and ϕ^2, in this order, and

$$S^{(+)} = \mathcal{R}^T(\vartheta) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{R}(\vartheta) = \begin{pmatrix} -\cos 2\vartheta & -\sin 2\vartheta \\ -\sin 2\vartheta & \cos 2\vartheta \end{pmatrix} , \tag{2.13}$$

where $\mathcal{R}(\vartheta)$ is the rotation matrix for angle ϑ. Finally the normalization constant is the g-factor [38] of the boundary state,

$$g^{(+)} \;=\; \sqrt{\frac{k_1^2 R_1^2 + k_2^2 R_2^2}{2 R_1 R_2}} \;=\; \frac{\ell}{\sqrt{2V}} \;=\; \sqrt{\frac{k_1 k_2}{\sin 2\vartheta}} \tag{2.14}$$

with ℓ the length of the D1-brane and V the volume of the torus. The last rewriting of the g-factor, which will be the most useful to us in the sequel, follows from simple trigonometric identities. The reader can easily verify that when $k_2 = \vartheta = 0$, the state (2.12) reduces to the tensor product of (2.4) with (2.5). The subscript "plus" refers to the sign of $-\det S^{(+)}$, or equivalently to minus the parity of the brane dimension. The relevance of this (seemingly upside-down) notation will become obvious in the following sections.

The other symmetric stable branes of the $c = 2$ theory can be obtained from the above D1 branes by a T-duality transformation of one of the scalars. With our conventions, the action of this transformation is[5]

$$R \to \frac{1}{2R} , \quad \tilde{a}_n \to -\tilde{a}_n , \quad \text{and} \quad (N, M) \to (M, N) . \tag{2.15}$$

T-dualizing one of the fields, say ϕ^1, maps $\|\mathrm{D1}, \vartheta\rangle\rangle$ to the boundary state

$$\|\mathrm{D2/D0}, \theta \,\rangle\rangle \;=\; \prod_{n=1}^{\infty} (e^{S^{(-)}_{ij} a^i_n \tilde{a}^j_n})^{\dagger} \left(g^{(-)} \sum_{N,M=-\infty}^{\infty} e^{iN\alpha' - iM\beta'} |k_1 M, k_2 N\rangle \otimes |-k_1 N, k_2 M\rangle \right) \tag{2.16}$$

[5]For a general discussion of the $O(d, d, Z)$ transformations of D-branes see [39].

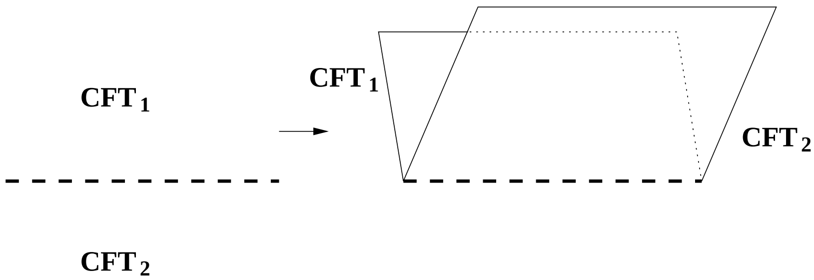

Figure 1: Folding trick.

where

$$S^{(-)} = S^{(+)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} , \quad \theta = \tan^{-1}\left(\frac{2k_2 R_1 R_2}{k_1}\right) , \tag{2.17}$$

and the g-factor of the brane is

$$g^{(-)} = \sqrt{\frac{k_1^2 + 4k_2^2 R_1^2 R_2^2}{4R_1 R_2}} = \sqrt{\frac{k_1 k_2}{\sin 2\theta}} . \tag{2.18}$$

Notice that since T-duality inverts $\sqrt{2}R_1$, the angle θ is in general not the same as ϑ. The two angles coincide only at the self-dual point of the radius R_1. The state (2.16) describes the bound state of k_1 D0s and k_2 D2s. As a check, note that for $(k_1, k_2) = (1, 0)$ or $(0, 1)$ one recovers the expressions of the pure D0, respectively the pure D2 brane. Note also that, consistently with our notation, $-\det S^{(-)} = -1$ and the dimension of the branes is even.

The generalization to oblique and to three-dimensional tori is straightforward. The boundary states for an oblique two-torus can be obtained by a sequence of T-duality transformations and rotations, starting with the elementary D0 brane. Furthermore, starting with the general D2/D0 brane on the (ϕ^1, ϕ^2) plane of a three-torus, one can rotate it to any other orientation in the compactification lattice. A T-duality then maps this to an arbitrary D3/D1 bound state. For $c = 4$ there exist new branes (e.g. along the Higgs branch of the D4/D0 system) which cannot be constructed by the above algorithm. We will not pursue the study of these higher-dimensional branes in the present work.

3. Unfolding and the topological maps

3.1 The unfolding procedure

A conformal interface between two theories, 1 and 2, can be mapped to a conformal boundary of the tensor-product theory $\text{CFT}_1 \otimes \text{CFT}_2$ by the folding trick shown in figure 1. Conversely, we can unfold a boundary state to an interface whenever the bulk CFT has two non-interacting components. Let us assume that in some appropriate basis, constructed by acting with self-adjoint (left and right) operators $\mathcal{O}_{\lambda_j}$ and $\mathcal{O}_{\tilde{\lambda}_j}$ on the vacuum, the boundary state takes the general form

$$\| \mathcal{B} \rangle\rangle \ = \ \sum \mathcal{B}_{\lambda_1 \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_2} \, |\lambda_1, \tilde{\lambda}_1\rangle \otimes |\lambda_2, \tilde{\lambda}_2\rangle \tag{3.1}$$

with coefficients $\mathcal{B}_{\lambda_1 \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_2}$ which are real.[6] We also assume that both CFTs are left-right symmetric. Then unfolding the boundary state (3.1) leads to the following interface and anti-interface, expressed as operators from the Hilbert space of CFT_2 to the Hilbert space of CFT_1 and vice-versa,

$$\begin{aligned} \mathcal{I}(1 \leftarrow 2) &= \sum \mathcal{B}_{\lambda_1 \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_2} \, |\lambda_1, \tilde{\lambda}_1\rangle\langle \tilde{\lambda}_2, \lambda_2| \,, \\ \mathcal{I}(2 \leftarrow 1) &= \sum \mathcal{B}_{\lambda_1 \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_2} \, |\lambda_2, \tilde{\lambda}_2\rangle\langle \tilde{\lambda}_1, \lambda_1| \,. \end{aligned} \tag{3.2}$$

Notice that unfolding flips the sign of the (closed-string) time coordinate τ for the unfolded theory, say CFT_2. It therefore involves both hermitean conjugation and the exchange of left- with right-movers, $\lambda_2 \leftrightarrow \tilde{\lambda}_2$. The individual matrix elements of the above operators are two-point functions on the sphere in the presence of the conformal interface.

Let us now specialize to the D1 and D2/D0 branes of the previous section. Since the torus is orthogonal and there is no B-field background, the two scalar fields are decoupled in the bulk, and the boundary states can be unfolded. Flipping the sign of τ in the expression (2.1) sends

$$\hat{N} \to -\hat{N}, \quad a_n \to -\tilde{a}_n^\dagger, \quad \text{and} \quad \tilde{a}_n \to -a_n^\dagger \ . \tag{3.3}$$

This is, as argued, hermitean conjugation followed by the exchange of left and right movers (the minus sign can be absorbed in the definition of basis). The only subtle point concerns the choice of a real basis of states. For the ground states, for example, one must work with the basis of states

$$\frac{|p, w\rangle + |-p, -w\rangle}{\sqrt{2}} \quad \text{and} \quad \frac{|p, w\rangle - |-p, -w\rangle}{\sqrt{2}i} \,, \tag{3.4}$$

in which the coefficients $\mathcal{B}_{\lambda_1 \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_2}$ are real. Hermitean conjugation followed by the reflection of momentum for the scalar ϕ^2, can then be shown to map $|p_2, w_2\rangle$ to $|-p_2, w_2\rangle$ in the expressions (2.12) and (2.16) of the boundary states. The final result for the interface operators therefore reads:

$$\mathcal{I}^{(\pm)\ (R_1 \leftarrow R_2)}_{(k_1,k_2)} \ = \ L^{(\pm)}_{(k_1,k_2)} \prod_{n=1}^{\infty} e^{\left(S^{(\pm)}_{11} (a^1_n)^\dagger (\tilde{a}^1_n)^\dagger - S^{(\pm)}_{12} (a^1_n)^\dagger a^2_n - S^{(\pm)}_{21} (\tilde{a}^1_n)^\dagger \tilde{a}^2_n + S^{(\pm)}_{22} a^2_n \tilde{a}^2_n \right)} \ , \tag{3.5}$$

where the ground state operators are the lattice sums:

$$L^{(+)}_{(k_1,k_2)}(\alpha, \beta) \ = \ \sqrt{\frac{k_1 k_2}{\sin 2\vartheta}} \times \sum_{N,M=-\infty}^{\infty} e^{iN\alpha - iM\beta} |k_2 N, k_1 M\,\rangle\langle k_1 N, k_2 M| \,, \tag{3.6}$$

$$L^{(-)}_{(k_1,k_2)}(\alpha, \beta) \ = \ \sqrt{\frac{k_1 k_2}{\sin 2\theta}} \times \sum_{N,M=-\infty}^{\infty} e^{iN\alpha - iM\beta} |k_1 M, k_2 N\,\rangle\langle k_1 N, k_2 M| \,, \tag{3.7}$$

[6] Put differently, the one-point functions of hermitean bulk operators on the disk must be real, an assumption that is certainly true for the toroidal branes considered here.

and in eq. (3.5) the daggered oscillators act implicitly on $L^{(\pm)}$ from the left. For the reader's convenience, we collect here also the expressions for $S^{(\pm)}$:

$$S^{(+)} = \begin{pmatrix} -\cos 2\vartheta & -\sin 2\vartheta \\ -\sin 2\vartheta & \cos 2\vartheta \end{pmatrix}, \quad \vartheta = \tan^{-1}(\frac{k_2 R_2}{k_1 R_1}), \tag{3.8}$$

$$S^{(-)} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \quad \theta = \tan^{-1}\left(\frac{2k_2 R_1 R_2}{k_1}\right). \tag{3.9}$$

eqs. (3.5) to (3.9) define the most general interfaces which separate two free-boson theories with radii R_1 and R_2, and which preserve a $\mathrm{U}(1) \times \mathrm{U}(1)$ subgroup of the $\mathrm{U}(1)^4$ symmetry of the bulk. Below we will refer to the + and the − operators as, respectively, even and odd. When no confusion is possible, their dependence on the phases (α, β) and on the radii $(R_1 \leftarrow R_2)$ will be omitted.

3.2 Reflection, transmission and entropy

It is important here to note that the operators (3.5) depend on the radii only through the angles ϑ and θ. This is true in particular for the matrices $S^{(\pm)}$, whose elements are the reflection and transmission coefficients across the interface [1, 13]. Total reflection occurs when ϑ or θ is a multiple of 90^o, which requires either k_1 or k_2 to vanish. This corresponds (up to Chan-Patton multiplicity) to the four factorizable boundary states of section 2, which unfold to the interface operators

$$\mathcal{I}_{\rm refl} \;=\; \| \, \mathrm{D}r_1 \rangle\!\rangle \langle\!\langle Dr_2 \| \quad \text{with} \quad r_1, r_2 = 0, 1 \; . \tag{3.10}$$

The two CFTs have in this case separate consistent boundaries, and they decouple completely.

More interesting is the case of total transmission, which occurs for angles that are an odd multiple of 45^o. The interface operators have now the form

$$\mathcal{I}^{(\pm)}_{\rm top} \;=\; L^{(\pm)} \prod_{n=1}^{\infty} e^{(-)^l \left[(a_n^1)^\dagger a_n^2 \pm (\tilde a_n^1)^\dagger \tilde a_n^2\right]}, \quad \text{for} \quad \vartheta, \theta = (2l+1)\frac{\pi}{4}. \tag{3.11}$$

It follows that the energy-momentum tensor is continuous across the interface, i.e. the Virasoro generators (not to be confused with the lattice sums!) obey the commutation relations

$$\mathrm{L}^1_n \, \mathcal{I}^{(\pm)}_{\rm top} \;=\; \mathcal{I}^{(\pm)}_{\rm top} \, \mathrm{L}^2_n \quad \text{and} \quad \widetilde{\mathrm{L}}^1_n \, \mathcal{I}^{(\pm)}_{\rm top} \;=\; \mathcal{I}^{(\pm)}_{\rm top} \, \widetilde{\mathrm{L}}^2_n \; . \tag{3.12}$$

Such interfaces have been dubbed topological , because they can be deformed freely across a Riemann surface. The topological interfaces for toroidal CFTs (both symmetric and non-symmetric) were analyzed recently in ref. [16]. Here we will only discuss a few, relevant for our purposes, features.

Consider first the case of defects, i.e. $R_1 = R_2 = R$. As argued generally by Fröhlich et al [9], the topological defects should include the generators of automorphisms of the CFT. At a generic value of the radius the only topological defects are

$$e(\alpha, \beta) \equiv \mathcal{I}^{(+)\;(R\leftarrow R)}_{(1,1)} \quad \text{and} \quad r(\alpha, \beta) \equiv \mathcal{I}^{(+)\;(R\leftarrow R)}_{(1,-1)} \; . \tag{3.13}$$

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These generate indeed the semidirect product $\mathrm{U}(1)^2 \rtimes Z_2$, i.e. the left and right translations and the reflections of ϕ. Notice that the trivial (identity) defect is $e(0,0)$, i.e. a diagonal D1-brane in the (ϕ^1, ϕ^2) plane, after folding. Turning on a Wilson line, translating and/or reflecting this diagonal D1-brane, gives all the other symmetry generators for generic R. At the self-dual radius, $2R_*^2 = 1$, there appear two new topological defects,

$$\omega \equiv \mathcal{I}^{(-)\ R_* \leftarrow R_*}_{(1,1)} \quad \text{and} \quad \tilde{\omega} \equiv \mathcal{I}^{(-)\ R_* \leftarrow R_*}_{(1,-1)} \ . \tag{3.14}$$

These generate T-duality twists, i.e. separate left and right reflections of ϕ. They enhance the symmetry to $(\mathrm{U}(1) \rtimes Z_2)^2$, which is the subgroup that preserves a maximal torus of the full $\mathrm{SU}(2)^2$ symmetry of the self-dual theory. The missing generators break more than two of the original $\mathrm{U}(1)$ symmetries of the free-scalar model, which explains why they were not included in our set of defects. Thus our analysis agrees with the observation of ref. [9].

What about other topological interfaces and defects? From the expression for the angles one sees that (provided $k_1 k_2 \neq 0$) every one of the operators (3.5) becomes topological at a special value of the ratio, or of the product of radii. Specifically this happens when

$$\frac{R_2}{R_1} = \left|\frac{k_1}{k_2}\right| \quad \text{or} \quad 2R_1R_2 = \left|\frac{k_1}{k_2}\right| \tag{3.15}$$

in the even, respectively odd case. Inspection of the lattice sums (3.6) and (3.7) reveals that when $|k_1k_2| > 1$ these operators map all but the proper sublattice $|k_1\mathbb{Z}, k_2\mathbb{Z}\rangle$ of the ground states to zero. The pairs of states that survive in these sums are states with equal $\mathrm{U}(1)$ charges and conformal weights. These higher topological interfaces do not therefore correspond to invertible operators, but rather to projectors, coupled with isomorphisms of appropriate subsectors of the two CFTs. For example, the $(2,1)$ topological interface maps the even-winding-number states of a theory at radius R to the even-momentum-number states of a theory at radius $2R$.

An important feature of topological interfaces is that they minimize the entropy, defined as the logarithm of the g factor, when the radii vary with (k_1, k_2) held fixed. This is a property reminiscent of the minimum-energy condition for BPS states. It is a simple consequence of the general expression for the g-function (ϑ must be replaced by θ in the odd case):

$$\log g \ = \ \log\sqrt{|k_1k_2|} - \log\sqrt{|\sin 2\vartheta|} \ . \tag{3.16}$$

The second contribution (which depends on the reflectivity [1, 13] of the interface) is non-negative, and it vanishes only in the topological case. The first, irreducible contribution is also non-negative, and it vanishes only for the symmetry-generating defects $(k_1, k_2) = (1, \pm 1)$. These latter are the only invertible maps, which is consistent with the fact that they should not generate any entropy. We conjecture that topological[7] interfaces between unitary conformal theories always have non-negative entropy, and that the bound is saturated only by CFT isomorphisms.

[7] More general interfaces can have a g factor smaller than one, and hence a negative entropy. An example is the totally-reflecting interface corresponding to a simple D2-brane, for which $g = \sqrt{R_1R_2}$. We thank Ingo Runkel and the JHEP referee for pointing this out.

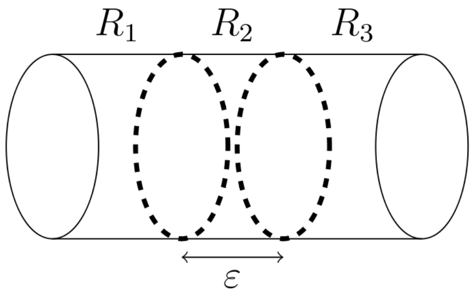

Figure 2: Fusion of two interfaces between three CFTs with radii R_1, R_2 and R_3. In the limit of vanishing separation, $\varepsilon \to 0$, the result should not depend on the value of the radius in the middle region.

The analogy of topological interfaces with BPS black holes can actually be pushed even further: one can interpret the topological conditions (3.15) as an attractor mechanism [32] that fixes the moduli of the bulk theory for any given set of "charges" $(k_1, k_2, \pm)$. Notice that there are two asymptotic regions and hence two bulk radii, but only one combination of the two is being fixed. Also, the entropy of a topological interface is the sum of the logarithms, rather than of the absolute values, of the integer charges. This is compatible with the fact that charges are multiplicatively conserved, as we are now going to explain.

4. The algebra of interfaces

4.1 Topological reduction of the fusion

Two boundary states, and hence also the corresponding interface operators, can be added. If they are identical, this amounts simply to introducing a Chan-Patton multiplicity. On the other hand, two operators can also be multiplied whenever the image of one lies in the domain of definition of the other. In the case at hand, this corresponds to juxtaposing an interface between CFT_1 and CFT_2 and an interface between CFT_2 and CFT_3, as shown in the figure 2. Because the product of these two operators is in general singular, we must first separate the interfaces by a distance ε. We work as before on the cylinder (σ, τ), so that the periodicity of σ sets the scale of distance. By the usual arguments of QFT we expect that the limit $\varepsilon \to 0$ can be rendered finite by a local self-energy counterterm. Accordingly, we define the fusion of the two interfaces as follows:

$$\mathcal{I} \circ \mathcal{I}' \;\equiv\; \lim_{\varepsilon \to 0} \; e^{2\pi d/\varepsilon}\, \mathcal{I}(1 \leftarrow 2)\, e^{-\varepsilon H_2}\, \mathcal{I}'(2 \leftarrow 3)\,, \tag{4.1}$$

where H_2 is the generator of τ-translations in CFT_2, and d/ε is the self-energy counterterm which must be adjusted so as to make the right-hand-side finite. Notice that this procedure is unambiguous because ε is the only relevant length scale in the problem.[8] The fact

[8] The inverse "temperature" 2π may only appear multiplicatively in the exponent.

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that $\mathcal{I}\circ\mathcal{I}'$ should be the sum of elementary $(1\leftarrow 3)$ interfaces with integer non-negative coefficients is a non-trivial check of the consistency of this definition.

Now the following intuitive but naive argument motivates one of the main points of this paper: in the limit $\varepsilon\to 0$ the region in the middle disappears, and so should any memory of the value of the radius in this region. Thus, modulo a local renormalization, the result should be independent of the value of R_2. To be more precise, given interfaces $\mathcal{I}^{(R_1\leftarrow R_2)}_{(k_1,k_2)}$ and $\mathcal{I}'^{(R_2\leftarrow R_3)}_{(k_1',k_2')}$ the fusion product is expected to be independent of variations in R_2, while all other quantities R_1, R_3, k_i, k_i' are held fixed. An additional argument is provided by continuity: if the fusion coefficients are integers they should not jump around as we vary R_2, except possibly at singular points in the moduli space. We will actually show that these naive arguments are correct in the case at hand, by computing explicitly (4.1) in the following section. Here we assume the result, and proceed to calculate the algebra.

This is made easy by the following trick: since the value of R_2 is irrelevant, we may choose it so as to make the interface $\mathcal{I}'$ topological. We have seen in the previous section that this is always possible, as long as $k_1'k_2'$ does not vanish.[9] Now using the commutation property of topological interfaces, eq. (3.12), we find

$$\mathcal{I}\, e^{-\varepsilon H_2}\mathcal{I}'_{\rm top} = \mathcal{I}\,\mathcal{I}'_{\rm top}\, e^{-\varepsilon H_3}, \tag{4.2}$$

and on the right-hand-side the $\varepsilon\to 0$ limit can be taken smoothly. Put differently, once $\mathcal{I}'$ has been made topological, it can be moved at no cost. We may thus restrict attention to non-singular products of one topological with one arbitrary interface.

The following observation simplifies the calculation even further: let us define the basic radius-changing interface, which is the deformed identity operator

$$e_{\rm def}^{(R_1\leftarrow R_2)} \equiv \mathcal{I}^{(+)\ (R_1\leftarrow R_2)}_{(1,1)} \quad \text{with} \quad \alpha=\beta=0\ . \tag{4.3}$$

Now an arbitrary conformal interface can be obtained as the product of a topological interface with this basic deformed identity. Explicitly:

$$\mathcal{I}^{(\pm)\ (R_1\leftarrow R_2)}_{(k_1,k_2)} \;=\; \mathcal{I}^{(\pm)\ (R_1\leftarrow R)}_{(k_1,k_2)}\ e_{\rm def}^{(R\leftarrow R_2)} \;= e_{\rm def}^{(R_1\leftarrow R')}\mathcal{I}^{(\pm)\ (R'\leftarrow R_2)}_{(k_1,k_2)}\ , \tag{4.4}$$

where R and R' are here implicitly adjusted so as to make the (k_1,k_2) operators topological. Let us prove the first equality, by evaluating explicitly the product in the even case and with $k_1k_2>0$ (the other three cases work in a similar way). From the general form (3.11) we see that the topological $(R_1\leftarrow R)$ operator commutes with the oscillator modes,

$$a_n^1\,\mathcal{I}^{(+)}_{\rm top} = \mathcal{I}^{(+)}_{\rm top}\,a_n \qquad\text{and}\qquad \tilde a_n^1\,\mathcal{I}^{(+)}_{\rm top} = \mathcal{I}^{(+)}_{\rm top}\,\tilde a_n\,, \tag{4.5}$$

where a_n and $\tilde a_n$ refer to the region of radius R, and the same equations hold for daggers. Thus in the expression (3.5) for the basic $R\leftarrow R_2$ interface we may replace the $a_n^\dagger$ by $(a_n^1)^\dagger$, and the $a_n^\dagger$ by $(a_n^1)^\dagger$. Furthermore the angle that enters the S-matrix of this interface is given by $\tan\vartheta = R_2/R = k_2R_2/k_1R_1$, where the second step uses the topological property

[9] If $k_1'k_2'=0$, then $\mathcal{I}'$ is totally reflecting and the CFT3 decouples. The problem reduces to the fusion of an interface with a boundary, a case that we will discuss in the end.

of the $R_1 \leftarrow R$ operator. Put differently, detaching a topological part does not change the reflectivity of the interface. Finally, the lattice sum in $e_{\rm def}$ is just the trivial isomorphism of momentum and winding states. Multiplying with the lattice sum of the $R_1 \leftarrow R$ operator completes the construction of the $R_1 \leftarrow R_2$ operator, and proves the relations (4.4) .

These relations show that all conformal interfaces (3.5) can be written as products of a deformed identity and a topological "dress", and that the latter can be moved off to the left or right. We can therefore calculate the product (4.2) by stripping $\mathcal{I}$ of its topological dress, multiplying this with the operator $\mathcal{I}'_{\rm top}$, and then dressing back the deformed identity on the left side. Hence, we need only to study the products of topological operators.

4.2 The multiplicative law for the charges

Since the oscillator modes enter in such products trivially, it is sufficient to multiply their lattice sums. Furthermore, by acting with the symmetry generator $e(\alpha, \beta)$ from the left or the right, we may set all phases in these lattice sums to zero. Notice that this is a non-commutative operation, e.g.

$$\mathcal{I}^{(+)}_{(k_1,k_2)}(\alpha,\beta) \;=\; e\left(\frac{\alpha}{k_2}, \frac{\beta}{k_1}\right) \mathcal{I}^{(+)}_{(k_1,k_2)}(0,0) \;=\; \mathcal{I}^{(+)}_{(k_1,k_2)}(0,0)\; e\left(\frac{\alpha}{k_1}, \frac{\beta}{k_2}\right) \tag{4.6}$$

with a similar equation for the odd case. When all the phases are set to zero, the product of two even topological operators reads:

$$\begin{aligned} &\sqrt{|k_1k_2k'_1k'_2|} \sum_{N,M,N',M'} |k_2N, k_1M\rangle\langle k_1N, k_2M \,||\, k'_2N', k'_1M'\rangle\langle k'_1N', k'_2M'| \\ &\qquad = \; JJ'\sqrt{|K_1K_2|} \sum_{N,M} |JK_2N, J'K_1M\rangle\langle\, JK_1N,\, J'K_2M\,|\,, \end{aligned} \tag{4.7}$$

where in the lower line we have redefined N and M so that they run unconstrained over all the integers, and

$$J = \gcd(k'_1, k_2)\,, \quad J' = \gcd(k_1, k'_2)\,, \quad K_1 = \frac{k_1k'_1}{JJ'}\,, \quad K_2 = \frac{k_2k'_2}{JJ'}\,, \tag{4.8}$$

with "gcd" standing for the greatest common divisor. If $J = J' = 1$, the product is just the elementary (K_1, K_2) interface. More generally, it is an array of JJ' such interfaces, arranged periodically in the (α, β) parameter space [16]. Periodic arrays couple indeed only to a sublattice of momenta and windings, as the reader will have no difficulty to verify. Explicitly, the product formula reads

$$\mathcal{I}^{(+)}_{(k_1,k_2)}(0,0) \circ \mathcal{I}^{(+)}_{(k'_1,k'_2)}(0,0) = \sum_{j,j'} \mathcal{I}^{(+)}_{(K_1,K_2)}\left(\frac{2\pi j}{J}, \frac{2\pi j'}{J'}\right), \tag{4.9}$$

where the sums run over $j = 1, \ldots J$ and $j' = 1, \ldots J'$.

This result can be expressed more elegantly if we mod out the action of the $\mathrm{U}(1)^2$ symmetry. Let us denote by $[k_1, k_2]^{(+)}$ the equivalence class of all D1-branes winding $(k_1, -k_2)$ times around the (ϕ^1, ϕ^2) torus. We also relax the condition that the winding

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numbers be relatively prime, but take note of the fact that the (open-string) moduli space has dimension equal to $2\gcd(k_1, k_2)$. For these equivalence classes of D-branes the fusion formula takes the simple form

$$[k_1, k_2]^{(+)} \circ [k_1', k_2']^{(+)} \;=\; [k_1k_1', k_2k_2']^{(+)} \;. \tag{4.10}$$

As anticipated already in section 3, the interface charges get multiplied and topological fusion conserves the entropy. Notice that the dimension of the moduli space of the product can be bigger than the sum of dimensions of its two factors. Thus a generic representative in the equivalence class on the right-hand side need not factorize. For example, three elementary $(1,1)$ interfaces can only be written in the product form $(1,3) \circ (3,1)$ if they are arranged in a periodic array.

To complete the derivation of the algebra, we need also to analyze the odd case. This can be done with the help of the T-duality twist, defined in eq. (3.14) at the self-dual point. Clearly this operator remains topological for any pair of radii such that $2R_1R_2 = 1$. A simple calculation shows that ω squares to 1, and that it exchanges the even and the odd interfaces as follows:

$$\omega \circ \mathcal{I}^{(-)}_{(k_1,k_2)}(\alpha,\beta) = \mathcal{I}^{(+)}_{(k_1,k_2)}(\alpha,\beta) \;, \quad \mathcal{I}^{(-)}_{(k_1,k_2)}(\alpha,\beta) \circ \omega = \mathcal{I}^{(+)}_{(k_2,k_1)}(-\beta,-\alpha)\,. \tag{4.11}$$

Together with equations (4.6) and (4.9), these twist relations are sufficient to compute the fusion of any two interfaces in the list (3.5). The final result, generalizing (4.10), can be worked out easily:

$$\begin{aligned} [k_1, k_2]^{(\pm)} \circ [k_1', k_2']^{(+)} \;&=\; [k_1k_1', k_2k_2']^{(\pm)} \,, \\ [k_1, k_2]^{(\pm)} \circ [k_1', k_2']^{(-)} \;&=\; [k_2k_1', k_1k_2']^{(\mp)} \;. \end{aligned} \tag{4.12}$$

A simple corollary of the above fusion rules is that the symmetry generators, together with one representative in the $[1,p]^{(+)}$ class for each prime number p, are sufficient to generate the entire algebra of these topological interfaces.

These fusion relations continue to hold for totally-reflecting interfaces, i.e. in the special case $k_1'k_2' = 0$. They then describe the action of the interface operators on the D-branes of the $c = 1$ model. For example, a Dirichlet condition in the left-half space corresponds to an operator in the class $[1,0]^{(-)}$ or $[0,1]^{(+)}$, where the two choices differ by a twist in the decoupled right-half region. Acting on this D0-brane with an operator in the $[k_1,k_2]^{(+)}$ class produces a periodic array of D0-branes, as illustrated in figure 3. This agrees with the simple geometric rule that was sketched in the introduction. All other actions of our interface operators on the D0-brane and the D1-brane of the $c = 1$ model can be obtained from this picture by T-duality twists. This completes our discussion of the algebra. We turn now to a proof of the argument that allowed the reduction to the topological case.

5. Entropy release and stability

5.1 Proof of the topological reduction

Let us return to the definition (4.1) of the fusion product. Using the relations (4.4) we can strip off the non-trivial topological parts, if any, of $\mathcal{I}$ and $\mathcal{I}'$ to the left, respectively right,

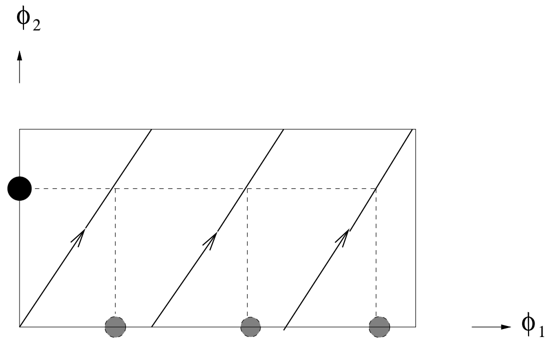

Figure 3: The action of a $[1,3]^{(+)}$ operator on the D0-brane of theory 2 (black dot) produces three D0-branes of theory 1 (light-colored dots). The latter are arranged periodically around the circle.

leaving in the middle two deformed identities, i.e. two basic radius-changing interfaces in the $(1,1)$ sector. This is illustrated in figure 4. The two radii, R_1' and R_3', in the nucleated regions are fixed by the requirement that the split-off interfaces be topological, as was explained in the previous section. We may now take the limit $\epsilon \to 0$, before dressing back the result with the topological interfaces from the left and right. To prove our claim, it is therefore sufficient to show that for any arbitrary triplet of radii we have

$$e_{\rm def}^{(R_1'\leftarrow R_2)} \circ e_{\rm def}^{(R_2\leftarrow R_3')} = e_{\rm def}^{(R_1'\leftarrow R_3')} \ , \tag{5.1}$$

i.e. that the product of deformed identities is the deformed identity. In the rest of this section we will explain why this is true. Readers not interested in the technical details can jump ahead to the next subsection.

Because the different frequencies of the scalar field do not talk, the calculation of the product of two interfaces factorizes into a separate calculation in each frequency sector. Thus the product, before taking the coincidence limit, reads

$$e_{\rm def}^{(R_1'\leftarrow R_2)}\ q^{H_2}\ e_{\rm def}^{(R_2\leftarrow R_3')} = \frac{1}{\sqrt{\sin 2\vartheta'\ \sin 2\vartheta}} \sum_{N,M} q^{h_{N,M}} |N,M\rangle\langle N,M| \prod_{n=1}^{\infty} O_n \ , \tag{5.2}$$

where $q \equiv e^{-\epsilon}$, $h_{N,M}$ is the energy of the (N,M) ground state in CFT$_2$, the operators O_n are the result of evaluating the product in the nth-frequency sector, and we have defined the angles

$$\tan\vartheta = \frac{R_2}{R_1'}\ , \quad \tan\vartheta' = \frac{R_3'}{R_2} \quad \text{and} \quad \tan\Theta = \frac{R_3'}{R_1'} = \tan\vartheta' \tan\vartheta \ . \tag{5.3}$$

Notice that, as we have stressed earlier, the topological dressing of an interface does not affect its angular orientation. The operator O_n is the product of the nth-frequency exponentials in the general expression (3.5) for a conformal interface, with q^{H_2} sandwiched in the middle, and with the whole thing evaluated in the ground state of CFT$_2$. The result depends on the oscillators $(a_n^1)^\dagger$, $(\tilde a_n^1)^\dagger, a_n^3$ and $\tilde a_n^3$ in the outer regions, as well as on the evolution parameter q and on the angles ϑ and ϑ'.

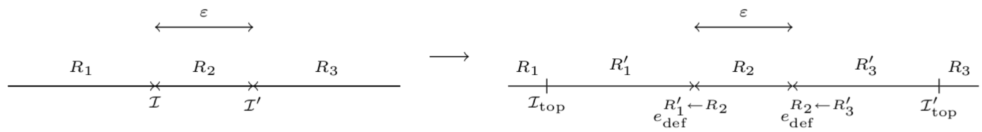

Figure 4: By stripping off their topological parts, we can relate the singularity in the fusion of any two conformal interfaces to the singularity in the product of two basic radius-changing operators.

To do this calculation, note that the oscillators of the outer regions can be treated effectively as c-numbers, and that the evolution operator can be absorbed into a rescaling of the daggered oscillators of the middle region,

$$(a_n^2)^\dagger \to q^n (a_n^2)^\dagger \quad \text{and} \quad (\tilde{a}_n^2)^\dagger \to q^n (\tilde{a}_n^2)^\dagger \ . \tag{5.4}$$

To lighten the notation, we will replace below $(\tilde{a}_n^2)^\dagger$ by $a^\dagger$ and similarly for the tilde oscillators. We also use the shorter notation $c \equiv \cos 2\vartheta$, $s \equiv \sin 2\vartheta$ and similarly for ϑ'. From eqs. (3.5) and (3.8) we can now read off the following expression for the operator O_n in the nth sector:

$$O_n = e^{B_1+B_3} \langle 0| e^{\left(c\,a\tilde{a} + aA_1 + \tilde{a}\tilde{A}_1\right)} \ e^{\left(-q^{2n}c' a^\dagger \tilde{a}^\dagger + q^n a^\dagger A_3 + q^n \tilde{a}^\dagger \tilde{A}_3\right)} |0\rangle \,, \tag{5.5}$$

where $|0\rangle$ is the ground state of the $(a, \tilde{a})$ system, and the capital letters stand for the following mutually-commuting operators:

$$\begin{aligned} A_1 &= s\,(a_n^1)^\dagger \,, & \tilde{A}_1 &= s\,(\tilde{a}_n^1)^\dagger \,, & B_1 &= -c\,(a_n^1)^\dagger (\tilde{a}_n^1)^\dagger \\ A_3 &= s'\,a_n^3 \,, & \tilde{A}_3 &= s'\,\tilde{a}_n^3 \,, & B_3 &= c'\,a_n^3 \tilde{a}_n^3 \ . \end{aligned} \tag{5.6}$$

We can calculate the matrix element in (5.5) by using the Gaussian representation, eq. (2.8), and commuting the order of the exponentials so that a passes to the right of $a^\dagger$. This is similar to the annulus calculation done in section 2. The result reads

$$O_n = e^{B_1+B_3} \int \frac{d^2z d^2w}{\pi^2} \, e^{-z\bar{z} - w\bar{w}} e^{(A_1 + cz)(q^n A_3 - q^{2n} c' w)} e^{(\tilde{A}_1 + \bar{z})(q^n \tilde{A}_3 + \bar{w})} \ . \tag{5.7}$$

Performing the Gaussian integrations over z and w and doing some straightforward algebra gives:

$$O_n \ = \ \frac{1}{1 + cc'q^{2n}} \exp\left[\begin{pmatrix} (a_n^1)^\dagger & \tilde{a}_n^3 \end{pmatrix} \begin{pmatrix} M_{11} & -M_{12} \\ -M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} (\tilde{a}_n^1)^\dagger \\ a_n^3 \end{pmatrix} \right]$$

with

$$M = \frac{1}{1 + cc'q^{2n}} \begin{pmatrix} -c - c'q^{2n} & ss'q^n \\ ss'q^n & c' + cq^{2n} \end{pmatrix} \ . \tag{5.8}$$

Plugging this result in eq. (5.2) gives the final expression for the product of two basic interfaces separated by a distance $\varepsilon = -\log q$.

We are now ready to take the $q \to 1$ limit. Simple trigonometry shows that in this limit M goes over smoothly to $S^{(+)}(\Theta)$, where Θ is the angle of the basic $(1 \leftarrow 3)$ interface.

Furthermore the lattice sum goes over smoothly to a multiple of the identity operator. Thus in the end

$$e_{\rm def}^{(R_1'\leftarrow R_2)} \circ e_{\rm def}^{(R_2\leftarrow R_3')} \;=\; \mathcal{N}\; e_{\rm def}^{(R_1'\leftarrow R_3')}\ , \tag{5.9}$$

where the normalization constant reads

$$\mathcal{N} \;=\; \lim_{\varepsilon\to 0}\;\; e^{2\pi d/\varepsilon}\, \sqrt{\frac{\sin 2\Theta}{\sin 2\vartheta\, \sin 2\vartheta'}}\; \prod_{n=1}^{\infty}(1+\cos 2\vartheta\, \cos 2\vartheta' q^{2n})^{-1}\,, \tag{5.10}$$

and d/ε is the divergent self-energy counterterm. To calculate the product in the limit, we use the Euler-MacLaurin formula:

$$-\sum_{n=1}^{\infty}\log(1+cc'e^{-2n\varepsilon}) = -\frac{1}{2\varepsilon}\int_0^1 \frac{dx}{x}\log(1+cc'x) + \frac{1}{2}\log(1+cc') + \cdots \tag{5.11}$$

The divergent part was first computed in ref. [1]. It is a Casimir energy, which is here removed by d/ε. The subtraction is, as we explained, unambiguous because ε is the only ultraviolet scale of the problem. The remaining finite terms combine nicely, with the help of the trigonometric identity

$$\sin 2\Theta \;=\; \frac{\sin 2\vartheta\, \sin 2\vartheta'}{1+\cos 2\vartheta\, \cos 2\vartheta'}\,, \tag{5.12}$$

to give $\mathcal{N}=1$. This completes the proof that the fusion is independent of the radius R_2 in the squeezed-in region, as advertized.

5.2 Decays of interfaces

A corollary of the above calculation is a universal formula for the entropy released in the fusion of two conformal interfaces. The result depends only on the angular orientations of the corresponding branes,

$$\begin{aligned}\Delta\log g \;&\equiv \log\big(g(\mathcal{I}\circ\mathcal{I}')\big) - \log\big(g(\mathcal{I})\big) - \log\big(g(\mathcal{I}')\big)\\ &= \frac{1}{2}\log(1+\cos 2\vartheta\, \cos 2\vartheta')\ .\end{aligned} \tag{5.13}$$

The sign of the entropy release is the same as the sign of the Casimir force d/ε^2, where $-d$ is the leading term in the expansion (5.11). Both are fixed by the product $(\cos 2\vartheta\, \cos 2\vartheta')$. When this product is negative $\mathcal{I}$ and $\mathcal{I}'$ tend to attract, and their entropy is lowered by fusion. This is in accordance with the prediction of the g-theorem [40]. Conversely, when Δg is positive the composite interface $\mathcal{I}\circ\mathcal{I}'$ is an unstable RG fixed point.

One can show that all non-topological interfaces are unstable[10] by the following argument: first strip off their non-trivial topological part, with the help of the dressing identities (4.4). What is left behind is the deformed identity operator, separating two regions with radii $R\neq R'$. Notice that these radii must be different, since otherwise the

[10] Their fusion with boundaries may, nevertheless, still produce stable D-branes.

original interface would be topological. Now the basic radius-jump interface is unstable to splitting into smaller jumps. Indeed, the dissociation

$$e_{\rm def}^{(R'\leftarrow R)} \;\; \rightarrow \;\; e_{\rm def}^{(R'\leftarrow R'')} \circ e_{\rm def}^{(R''\leftarrow R)} \tag{5.14}$$

is entropically favoured whenever $R < R'' < R'$ or $R' < R'' < R$. This follows directly from (5.13). The same conclusion could in fact be reached by considering the effective theory for the radius field, $\mathcal{L} \sim (\partial R/R)^2$, in which domain walls tend to spread to infinite thickness. This splitting-off of radius jumps tends to push the bulk radii to their attractor fixed values. Conformal interfaces could thus prove to be a useful tool for studying the coupled bulk and boundary RG flows in string theory.[11]

What about the topological interfaces, whose fusion generates no entropy? These are marginally unstable against decay to 'prime-factor partons', i.e. $(1,p)$ or $(p,1)$ interfaces with p a prime number. In the case of BPS black holes the analogous decays are hindered by infinite-throats [42], so that the bound and unbound states can be distinguished. In the case at hand, recombination generally increases the dimension of the open-string moduli space and it is unclear whether such a distinction makes sense. Notice that there is no process which can reduce the entropy of the $(1,p)$ "partons" back to zero. Annihilation with the "antiparton" $(p,1)$ releases an entropy $\log p$.

6. Quantum interfaces

The interfaces discussed so far connect two points in the S^1 moduli-space of the $c = 1$ models. For more general interfaces, the conformal theories on the two sides live in different branches of moduli space, or may even be completely different theories. In the latter case, it has been shown by Quella et al [13] that the difference of the two central charges, $|c_1 - c_2|$, provides a lower bound to the reflectivity of the interface. Such interfaces are thus never topological, and may be unstable against dissociation processes like those discussed in the previous section. This is an interesting question that we will not address here.

Let us consider instead the topological interfaces that connect the circle with the orbifold branch. Examples of such interfaces are easy to construct. They include all D1-branes on $S^1 \times (S^1/Z_2)$ with a 45^o orientation. To be more specific, consider the interfaces on the circle line, setting $\alpha = \beta = 0$ for simplicity, and with $k_1 = 2l_1$ even. Then the linear combinations

$$(2l_1, |k_2|)_{\rm cir/orb} \;\equiv\; \frac{1}{2}(2l_1, k_2)^{(+)} + \frac{1}{2}(2l_1, -k_2)^{(+)} \tag{6.1}$$

are good conformal interfaces connecting the circle and the orbifold branch. Note that half-integer coefficients would have been forbidden for an interface between circle theories. They are here admissible because D1-branes and their images under ϕ^2 reflection are identified. As a concrete example consider the $(2,1)_{\rm cir/orb}$ interface. It becomes topological when the radius of the orbifold is double that of the circle. Inspection of the lattice sum (3.6) shows that this interface projects out the odd-winding sectors of the circle theory, and the odd-momentum and twisted sectors of the orbifold theory. It identifies in an obvious manner

[11] Coupled bulk and boundary RG flows have been studied differently in ref. [41].

the remaining states. The entropy of the map is $\log g = \log\sqrt{2}$. Many other circle/orbifold and orbifold/orbifold interfaces can be written down in a similar way.

These and all previous topological interfaces share one important common feature: they have a bulk (closed-string) modulus, which is the product or the ratio of radii on the two sides. Correspondingly, there is a semi-classical regime where their action is, modulo a T-duality transformation, geometrical. For the even interfaces, for example, the classical regime is the limit of large radii with the ratio $R_1/R_2 = |k_2/k_1|$ kept fixed. It is well known, on the other hand, that there exist many non-geometric D-branes, and the same is true for conformal interfaces. For instance, when $R_{\rm orb} = 2R_{\rm cir} = R_*$ the orbifold and the circle theories are the same [43], so there exists an isomorphism, τ, between the two. It is certainly not contained in the list (6.1) because it has zero entropy. When composed with topological maps, from the circle and/or from the orbifold side, it generates a whole new class of interfaces, with both the circle and the orbifold radius fixed. We may refer to such non-geometric interfaces, deep in the CFT moduli space, as purely 'quantum'.

Quantum interfaces actually exist also on the circle line. They are generated by the enhanced $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$ isometries of the theory at the self-dual radius, as discussed in ref. [16]:

$$e^{(R_*\leftarrow R_*)}(h,\tilde{h}) \qquad \text{for all} \quad h,\tilde{h}\in \mathrm{SU}(2)\ . \tag{6.2}$$

Multiplying these isometries with our topological operators, from left and right, gives a large class of topological interfaces:[12]

$$\mathcal{I}^{(R_1\leftarrow R_2)}(h,\tilde{h};k_1,k_2;k_1',k_2') \ \equiv\ \mathcal{I}^{(+)}_{(k_1,k_2)} \circ e^{(R_*\leftarrow R_*)}(h,\tilde{h}) \circ \mathcal{I}^{(+)}_{(k_1',k_2')}\ . \tag{6.3}$$

For these to be topological both radii must be a priori fixed,

$$R_1 = \left|\frac{k_2}{k_1}\right| R_* \quad \text{and} \quad R_2 = \left|\frac{k_1'}{k_2'}\right| R_* \ . \tag{6.4}$$

The above operators reduce, in fact, to the even and odd interfaces of the previous sections when h and $\tilde{h}$ commute (up to reflection) with the $\mathrm{U}(1)^2$ generators of the circle line. In this special situation, the constraint on one combination of radii gets relaxed. For more general rotations, these interfaces break all $\mathrm{U}(1)$ symmetries of the model. As shown in ref. [16] , their action on the basic D-branes of the circle theory produces the continuous extrapolations between arrays of D0 and D1 branes that were constructed in ref. [31]. This shows that topological defects can act on D-branes in non-trivial ways.

It would be very interesting to extend the analysis of our paper to these, purely quantum, generators of the interface algebra. This is not straightforward, because it is unclear how to separate these topological dresses in a fusion process. The results of ref. [41] actually suggest that the radius deformations of the enlarged algebra may be singular. We hope to return to these questions in the near future.

[12] Odd interfaces do not give new operators, because the duality twist is a special $\mathrm{SU}(2)_r$ isomorphism. Chains of topological operators between two twists also do not produce new operators. Such chains can be always fused to give an operator with $k_1 = k_2$, which can then be written as a superposition of symmetry generators.

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Acknowledgments

We thank Matthias Gaberdiel, Bernard Julia, Daniel Roggenkamp and Jan Troost for discussions. This work was supported by a EURYI award of the European Science Foundation (I.B.), and by the European Networks 'Superstring Theory' (MRTN-CT-2004-512194) and 'Forces Universe' (MRTN-CT-2004-005104).

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